Tutorial 9 : Selected problems of Assignment 9

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21/11/2018

Recall the Initial Value Problem:

Def An Initial Value Problem consists of the following

$$\begin{cases} \frac{dx}{dt} = f(t,x) \\ x(t_0) = x_0 \end{cases} \qquad \text{(IVP)}$$

where $f: R := [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] \to \mathbb{R}$ is a function.

An IVP is (uniquely) solvable for $a' \in (0,a)$ if there exists (unique)

$x(t): \underbrace{[t_0 - a', t_0 + a']}_{I_{a'}(t_0)} \longrightarrow \underbrace{[x_0 - b, x_0 + b]}_{I_b(x_0)}$ s.t. $x(t)$ is C^1 and solves IVP:

$$\begin{cases} x'(t) = f(t,x), \quad \forall\, t \in I_{a'}(t_0) \\ x(t_0) = x_0 \end{cases}$$

Thm (Picard-Lindelöf) Given an IVP as above,

① If $f \in C(R)$ satisfies a Lipschitz condition (uniform in t), i.e.

$$\exists L > 0 \text{ s.t. } \forall (t, x_1), (t, x_2) \in R, \quad |f(t,x_1) - f(t,x_2)| \le L|x_1 - x_2|$$

then IVP is uniquely solvable for $0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}$, $M := \sup_R |f(t,x)|$ (assuming $M > 0$)

② If in addition $f \in C^k(R)$, $\exists k \ge 1$, then $x(t) \in C^{k+1}(I_{a'}(t_0))$

Q1) (HW9, Q5) Using the perturbation of identity, prove ①

for $0 < a' < \min\{a, \frac{b}{M_0 + Lb}, \frac{1}{L}\}$, where $M_0 := \sup_{t \in I_a(t_0)} |f(t, x_0)|$

Pf) Recall that by Prop. 3.11, it suffices to solve the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds, \text{ where } x: I_{a'}(t_0) \to I_b(x_0) \text{ is continuous.}$$

Equivalently: $x(t) + \left(-x_0 - \int_{t_0}^{t} (f(s, x(s)) - f(s, x_0))ds\right) = \int_{t_0}^{t} f(s, x_0) ds$

Applying the perturbation of identity with $(X, \|\cdot\|) = (C[t_0 - a', t_0 + a'], \|\cdot\|_\infty)$

$\Phi: X \to X$ by $\Phi(x(t)) = x(t) + \left(-x_0 - \int_{t_0}^{t} (f(s, x(s)) - f(s, x_0))ds\right)$

$= (I + \Psi)(x(t))$, where $\Psi(x(t)) = -x_0 - \int_{t_0}^{t} (f(s, x(s)) - f(s, x_0))ds$

Let $x_0(t), y_0(t) \in X$ be defined as $\begin{cases} x_0(t) = x_0, & \forall t \in I_{a'}(t_0) \\ y_0(t) = 0 \end{cases}$

then $\Phi(x_0(t)) = y_0(t)$.

Checking $\Psi: X \to X$ is a contraction: $\forall x_1(t), x_2(t) \in X$, $\forall t \in I_{a'}(t_0)$

$|\Psi(x_1(t)) - \Psi(x_2(t))| = \left| \int_{t_0}^{t} (f(s, x_1(s)) - f(s, x_2(s)))ds \right| \leq \int_{t_0}^{t} L \cdot |x_1(s) - x_2(s)|\, ds$

$\leq L \cdot \|x_1 - x_2\|_\infty |t - t_0| \leq (La') \cdot \|x_1 - x_2\|_\infty = \gamma \|x_1 - x_2\|_\infty$, where $\gamma := La' < 1$

$\therefore \|\Psi(x_1) - \Psi(x_2)\|_\infty \leq \gamma \|x_1 - x_2\|_\infty$

$\therefore$ By the perturbation of identity, $\forall r>0$, $R := (1-\gamma)r$,

$\forall y(t) \in \overline{B_R(y_0(t))}$, $\exists!\, x(t) \in B_r(x_0(t))$ s.t. $\Phi(x(t)) = y(t)$

In particular, choose $r=b$, $R=(1-La')b$,

Checking $y(t) := \int_{t_0}^{t} f(s,x_0)\,ds \in B_R(y_0(t))$: $\forall t \in I_{a'}(t_0)$,

$$|y(t)-y_0(t)| = \left|\int_{t_0}^{t} f(s,x_0)\,ds\right| \leq M_0 \cdot |t-t_0| \leq M_0 \cdot a' < (1-La')b = R$$

$\left(\text{since } a' < \frac{b}{M_0+Lb} \Leftrightarrow M_0 a' + Lba' < b \Leftrightarrow M_0 a' < (1-La')b\right)$

Therefore, $\exists!\, x(t) \in \overline{B_b(x_0(t))}$ s.t. $\Phi(x(t)) = \int_{t_0}^{t} f(s,x_0)\,ds$

i.e. $\exists!\, x(t) : I_{a'}(t_0) \to I_b(x_0)$ satisfying the integral equation.

$-\blacksquare$

Q2) (HW9, Q6) Prove ②.

Sol) Induction on $k \geq 0$: $f \in C^k(R) \Rightarrow x(t) \in C^{k+1}(I_{a'}(t_0))$

$k=0$: Follows from the integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, dt$

by the Fundamental Theorem of Calculus (FTC).

Suppose the statement holds for $k = K$, then for $k = K+1$,

assume $f \in C^{k+1}(R)$, then $f \in C^k(R)$, hence by Inductive hypothesis

$x(t) \in C^{k+1}(I_{a'}(t_0))$. Therefore, $f(t, x(t)) : I_{a'}(t_0) \to \mathbb{R}$ is C^{k+1}.

then $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \in C^{k+2}(I_{a'}(t_0))$ by FTC

$\therefore$ By Induction, $\forall k \geq 0$, $f \in C^k(R) \Rightarrow x(t) \in C^{k+1}(I_{a'}(t_0))$ — $\blacksquare$

Q3) (HW9, Q9) Show that the following integral equation

$$h(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2} h(y)\,dy \text{ , where } h \in C[-1,1]$$

has a unique solution. Moreover, show that h is nonnegative.

Sol: Define $(X,d) = (\{h \in C[-1,1] \mid h(x) \geq 0, \forall x \in [-1,1]\}, \|\cdot\|_\infty)$

Define $T: X \to C[-1,1]$ by $(Th)(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2} h(y)\,dy$

then $\forall x \in [-1,1]$, $(Th)(x) \geq 1 > 0$. $\therefore$ $T: X \to X$

Also, T is a contraction: $\forall h_1, h_2 \in X$, $\forall x \in [-1,1]$,

$$|(Th_1)(x) - (Th_2)(x)| = \left| \frac{1}{\pi} \int_{-1}^{1} \frac{1}{1+(x-y)^2} (h_1(y) - h_2(y))\,dy \right|$$

$$\leq \frac{1}{\pi} \cdot 1 \cdot \|h_1 - h_2\|_\infty \cdot (1-(-1)) = \frac{2}{\pi} \|h_1 - h_2\|_\infty$$

$\therefore$ $\|Th_1 - Th_2\| \leq \gamma \|h_1 - h_2\|_\infty$, where $\gamma = \frac{2}{\pi} < 1$

$\therefore$ By Contraction Mapping Principle, T has a unique fixed point.

i.e. the above integral equation has a unique non-negative solution. $\square$